Prolate Spheroidal Wave Functions, Quadratures, Interpolation, and Applications

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Motivation

- $\triangleright$ Suppose that f vanishes outside $[-T, T]$:

$$F(w) = \int_{-T}^{T} f(t) e^{-iwt} dt\,.$$

- $\triangleright$ Suppose that $\bar{f}$ results from lowpass filtering:

$$\bar{f}(t) = \frac{1}{2\pi} \int_{-c}^{c} F(w) e^{iwt} dw\,.$$

Which $f \in L^2(-\infty, \infty)$ loses the smallest fraction of energy, that is, which f maximizes

$$\mu = \frac{\int_{-\infty}^{\infty} |\bar{f}(t)|^2 dt}{\int_{-\infty}^{\infty} |f(t)|^2 dt} \qquad ?$$

- $\triangleright$ To ask the question differently: what time-limited function f minimizes

$$S = \frac{||F||^2_{(-\infty,+\infty)}}{||F||^2_{[-c,c]}}$$

(supergain)

Band-limited functions that are also time-limited?

Structure of the Talk

- ▷ Introduction of Band-limited Functions
- ▷ Prolate Spheroidal Wave Functions, History, Subject of Our Work
- ▷ Classical Theory in Modern Language
- ▷ Numerical Algorithms: Quadratures, Interpolation
- ▷ Formulae for Certain Special Values
- ▷ An Application

Introduction: Band-Limited Functions

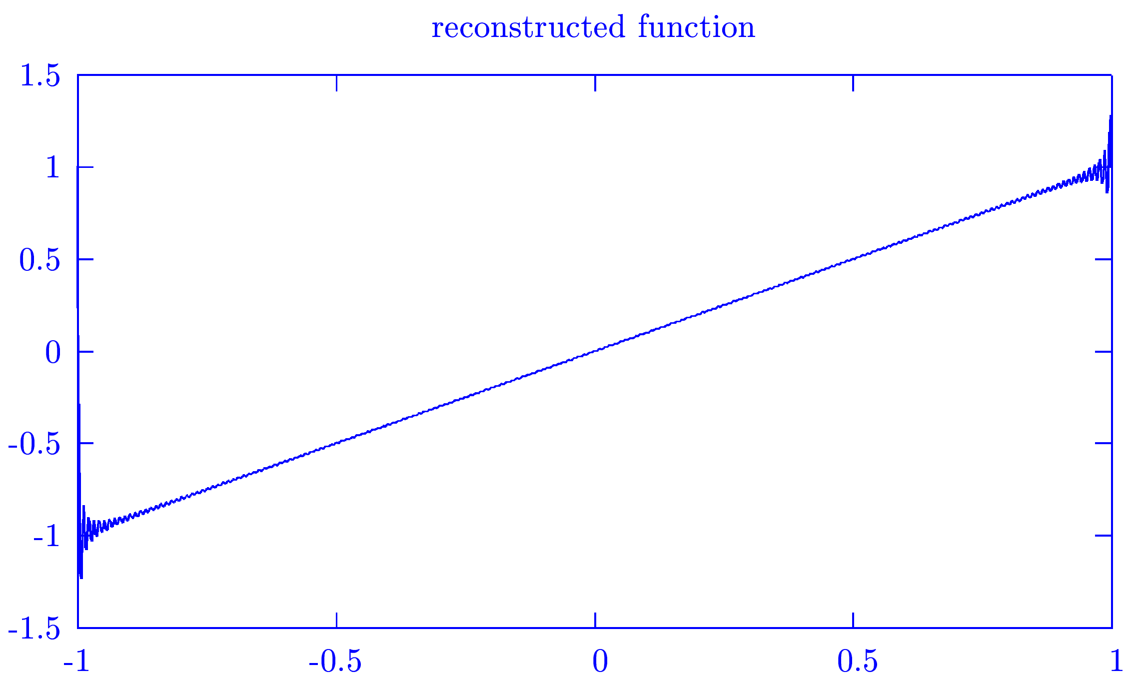
- Band-limited functions

$$f(x) = \int_{-1}^{1} e^{icxt}\,\sigma(t)\;dt.$$

- Fourier transform with compact support
- Examples: $\sin(m \cdot t)$, $\sin(t) + \cos(3.1\,t)$, $I_1 + I_2 + 2\sqrt{I_1 \cdot I_2}\,\cos(\phi_2 - \phi_1)$
- Ubiquitous: wave phenomena, measurements, engineering problems
- Importance was recognized 150 years ago, Fourier analysis

Fourier Methods, Gibbs Phenomenon

- Reconstruction of Discrete Fourier Transform for $f(x) = x$ on $[-1, 1]$.



- Jump discontinuity at the ends of the interval
- Fourier methods work well when functions have “smooth” periodic extensions on the entire real line

Prolate Spheroidal Wave Functions

- ▷ Band-limited and “time-concentrated”
- ▷ Initially known as solution to the second order ordinary differential equation

$$((1 - x^2)\,\psi'(x))' + (\chi - c^2 x^2)\,\psi(x) = 0$$

- ▷ Studied as special functions in mathematical physics (around 1850)
- \- Various evaluation schemes: expansions based on polynomials, Bessel functions, Weber functions, etc. (1880 - 1940)
- ▷ Classical evaluation scheme, three-term recursion, Bouwkamp (1942)
- ▷ Unstable for large-scaled problems

Prolate Spheroidal Wave Functions (continued)

- Differential operator

$$L(\psi) = ((1 - x^2)\, \psi'(x))' - c^2 x^2\, \psi(x)$$

  and integral operator

$$Q(\psi) = \int_{-1}^{1} \psi(t)\, e^{icxt}\, dt$$

  commute!

- Analytical properties, applications in electrical engineering by Slepian and colleagues at Bell Laboratories (1960s)
- Sequences of famous papers; not used as a numerical tool
- Applications in antenna design by Rhodes (1974)
- Limited by the availability of PSWFs

Mathematical Properties

Sturm-Liouville Eigenvalue Problem

$$((1-x^2)\,\psi_n'(x))' - c^2x^2\,\psi_n(x) + \chi_n\,\psi_n(x) = 0$$

- ▷ For each $c > 0$, eigenvalues χ_n^c are positive, can be ordered in increasing order; ψ_n^c is the corresponding n-th order eigenfunction
- ▷ $\{\psi_n^c\}$ form a basis for $L^2[-1,1]$ functions, with weight function 1
- ▷ ψ_n^c is real-valued
- ▷ ψ_{2n}^c are even, and ψ_{2n+1}^c are odd
- ▷ ψ_n^c has n real and simple roots on [-1, 1]
- ▷ Scaling, orthonormal basis for $L^2[-1,1]$

$$\int_{-1}^{1} \psi_n^c(t) \cdot \psi_m^c(t)\; dt = \delta_{m,n}$$

Mathematical Properties (continued)

- ▷ Eigenfunctions

$$\int_{-1}^{1} \frac{\sin c(t-x)}{\pi(t-x)} \cdot \psi_n^c(t)\ dt = \lambda_n^c \cdot \psi_n^c(x)$$

- ▷ (surprise!) Orthogonal on R^1

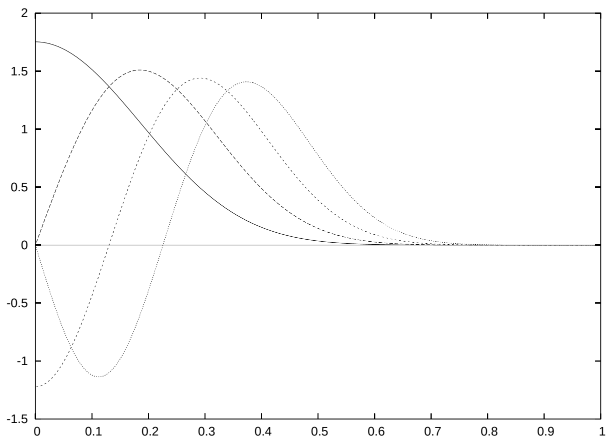
$$\int_{-\infty}^{\infty} \psi_n(t) \cdot \psi_m(t)\ dt = \frac{\delta_{m,n}}{\lambda_m}$$

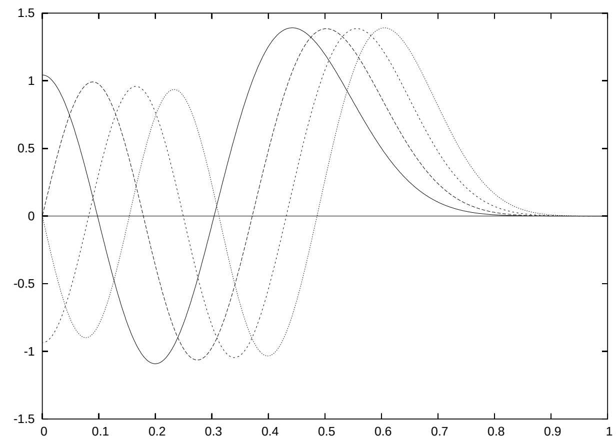
- ▷ (obviously?) $\{\psi_n^c\}$ form a basis for functions of band-limit c on R^1

- ▷ Analytic on C, a rich collection of identities

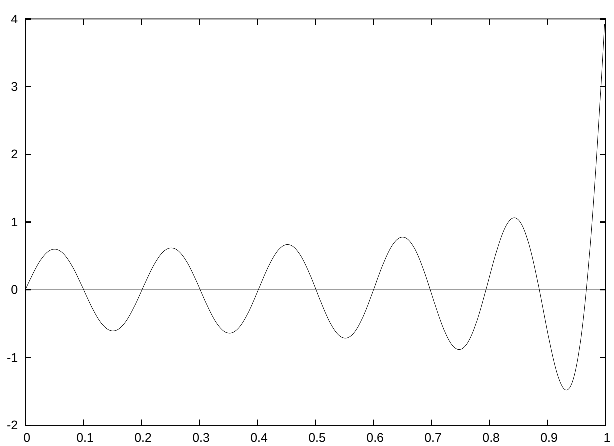
Examples of PSWFs

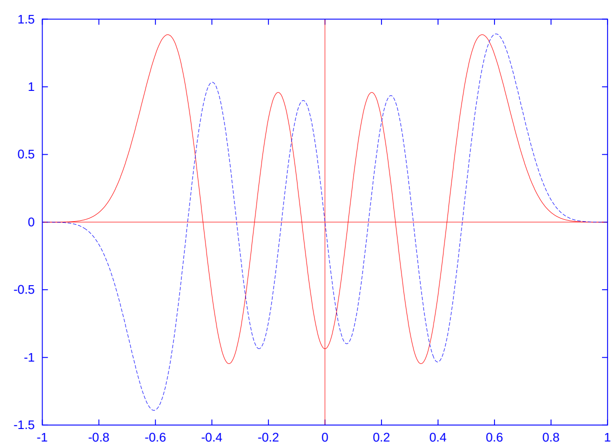
Prolate Functions #0 - #3 and #4 - #7 ($c = 30$)





Prolate Functions #30, even and odd PSWFs





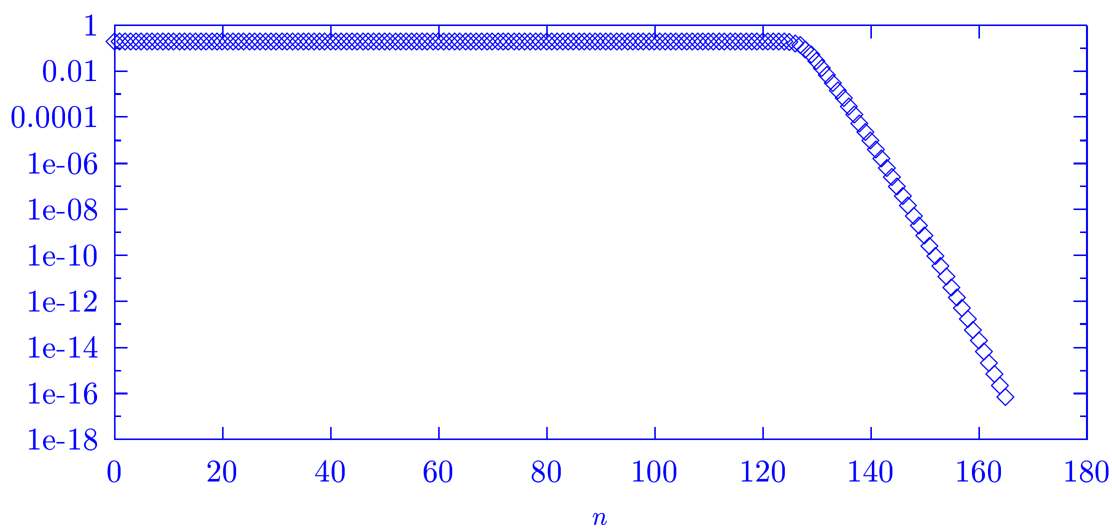
“Supergain”

$$\int_{-\infty}^{\infty} (\psi_n^c(x))^2 \; dx = \frac{1}{\lambda_n^c},$$

or

$$\frac{\int_{-\infty}^{\infty} (\psi_n^c(x))^2 \; dx}{\int_{-1}^{1} (\psi_n^c(x))^2 \; dx} = \frac{1}{\lambda_n^c},$$

▷ Behavior of λ_n^c, $\frac{2c}{\pi}(\approx 127)$
λ_n^c vs. n for $c = 200$:



▷ Qualitative discussion of the implied behavior of ψ_n^c on R^1

Classical Theory: Connection to Legendre Polynomials

- ▷ Legendre Polynomials satisfy

$$((1-x^2)\,P_n'(x))' + n(n+1)\,P_n(x) = 0$$

- ▷ Three-term Recursion

$$P_{n+1}(x) = \frac{2n+1}{n+1}\,x\,P_n(x) - \frac{n}{n+1}\,P_{n-1}(x)$$

- ▷ Prolate functions satisfy

$$((1-x^2)\,\psi_n'(x))' + (\chi_n - c^2\,x^2)\,\psi_n(x) = 0$$

- ▷ Expanding Prolate functions in Legendre Series

$$\psi_m(x) = \sum_{n=0}^{\infty} \alpha_n^m \cdot P_n(x)$$

- ▷ Three-term recursion

$$a_n \cdot \alpha_{n+2}^m + b_n \cdot \alpha_n^m + c_n \cdot \alpha_{n-2}^m = 0\,,$$

with

$$a_n = \frac{(n+2)(n+1)}{(2n+3)\sqrt{(2n+5)(2n+1)}} \cdot c^2$$

$$b_n = n(n+1) + \frac{2n(n+1)-1}{(2n+3)(2n-1)} \cdot c^2 - \chi_m$$

$$c_n = \frac{n(n-1)}{(2n-1)\sqrt{(2n-3)(2n+1)}} \cdot c^2$$

- ▷ b_n is dominant for large n
- ▷ ψ_m is smooth, and α_n^m decay superalgebraically (once $n > c$)

Tri-diagonal Matrix

$$\begin{pmatrix} b_0 & a_2 & 0 & \dots & \dots \\ c_2 & b_2 & a_4 & 0 & \dots \\ 0 & c_4 & b_4 & a_6 & \dots \\ & 0 & c_6 & b_6 & \dots \\ & & \vdots & & \ddots \end{pmatrix} \cdot \begin{pmatrix} \alpha_0^m \\ \alpha_2^m \\ \alpha_4^m \\ \alpha_6^m \\ \vdots \end{pmatrix} = \chi_m \cdot \begin{pmatrix} \alpha_0^m \\ \alpha_2^m \\ \alpha_4^m \\ \alpha_6^m \\ \vdots \end{pmatrix}$$

- ▷ Symmetric, diagonally dominant for large n
- ▷ χ_m are eigenvalues
- ▷ Standard QR scheme for χ_m
- ▷ We used Wilkinson's subroutine published in 1964
- ▷ Bouwkamp did not have it
- ▷ Legendre coefficients α_n^m are coordinates of eigenvectors
- ▷ Coefficients $\alpha_0^m, \alpha_2^m, \alpha_4^m, \dots$ and $\alpha_1^m, \alpha_3^m, \alpha_5^m, \dots$ can be computed separately with, for example, Inverse Power Method

Numerical Evaluation of $\psi(x)$

- Generate the leading n rows and columns of A:

$$n > \frac{2c}{\pi} + \left(\frac{1}{\pi^2}\log\frac{1}{\varepsilon}\right)\log(c) + 10\cdot\log(c)$$

(Fuchs 1964)

- Obtain eigenvalues and eigenvectors for the symmetric tri-diagonal matrices using standard numerical subroutines
- Evaluate ψ_m using its Legendre expansion
- Essentially Bouwkamp algorithm in modern language, straightforward and robust
- Cost: $O(c^2)$ operations for computing the coefficients ($O(c\,n)$ operations for computing the coefficients for the first n Prolate functions, n is proportional to c)
- $O(c)$ operations per subsequent evaluation

Quadrature and Interpolation for Band-Limited Functions

- Deal with band-limited functions on R^1

$$f^c(x) = \int_{-1}^{1} \sigma(t)\, e^{icxt}\, dt$$

- Sums of the form

$$\sum_{m=0}^{N} d_m\, \psi_m^c(x)$$

  - Similar to polynomials : number of roots on $[-1, 1]$, division theorem, etc
  - Roots of $\psi_m^c(x)$ are quadrature nodes (!)

- Remarkably similar to Gaussian quadratures of polynomials, positive weights, symmetry, efficiency

- Accuracy is roughly λ_N

Construction of Quadratures

- Division Theorem:

$$f^{2c}(x) = \psi_n^c(x)\, q^c(x) + r^c(x)$$

- Choosing nodes x_i as roots of $\psi_n^c(x)$, we have

$$\int_{-1}^{1} f^{2c}(x)\ dx = C \cdot \lambda_n + \int_{-1}^{1} r^c(x)\ dx$$

and

$$\sum_{i=0}^{n} w_i\, f^{2c}(x_i) = 0 + \sum_{i=0}^{n} w_i\, r^c(x_i)$$

- Weights w_i: solve the linear system

$$\begin{aligned}
\sum_{i=1}^{n} w_i\, \psi_0^c(x_i) &= \int_{-1}^{1} \psi_0^c(x)\ dx \\
\sum_{i=1}^{n} w_i\, \psi_1^c(x_i) &= \int_{-1}^{1} \psi_1^c(x)\ dx \\
&\vdots \\
\sum_{i=1}^{n} w_i\, \psi_{n-1}^c(x_i) &= \int_{-1}^{1} \psi_{n-1}^c(x)\ dx
\end{aligned}$$

Interpolation Algorithm

Given functions of band-limit $2c$, and given the expected precision ϵ

- ▷ Find n, such that the norm of eigenvalue $\lambda_n^c < \epsilon$
- ▷ Compute nodes x_i as roots of ψ_n^c;
- ▷ Construct weights w_i by solving the linear system

$$
\begin{aligned}
\sum_{i=1}^{n} w_i \, \psi_0^c(x_i) &= \int_{-1}^{1} \psi_0^c(x) \, dx \\
\sum_{i=1}^{n} w_i \, \psi_1^c(x_i) &= \int_{-1}^{1} \psi_1^c(x) \, dx \\
&\vdots \\
\sum_{i=1}^{n} w_i \, \psi_{n-1}^c(x_i) &= \int_{-1}^{1} \psi_{n-1}^c(x) \, dx
\end{aligned}
$$

Accuracy vs. the Number of Nodes

- ▷ Unlike the case of polynomials, accuracy is limited: the roots of ψ_n^c provide an accuracy of roughly λ_n .
- ▷ $\lambda_n < \varepsilon$ for all (approximately)

$$n > \frac{2c}{\pi}$$

- ▷ For large c, n is almost independent of ε

Quadrature performance for varying band limits, for $\varepsilon = 10^{-7}$

c	n	nodes/λ	Error	N_{Gauss}
10.0	9	2.827	0.51E-07	13
50.0	24	1.508	0.83E-07	37
90.0	38	1.326	0.40E-07	59
200.0	74	1.162	0.86E-07	118
600.0	203	1.063	0.11E-06	326
800.0	267	1.049	0.13E-06	428
1000.0	331	1.054	0.14E-06	530
1800.0	587	1.025	0.80E-07	937
2400.0	778	1.018	0.15E-06	1240
4000.0	1288	1.012	0.17E-06	2047

Prolate vs. Gaussian

Tested for $\sin(a \cdot x)$ where $a \in [-10, 10]$, $x \in [-1, 1]$. Quadratures were constructed with the same number of nodes, and tested in double-precision arithmetics.

n	Gaussian Error	Prolate Error
5	2.9d-15	2.2d-15
9	2.4d-15	3.6d-15
16	4.7d-15	1.8d-15
36	3.9d-14	1.8d-15
101	6.1d-13	5.2d-15
350	3.0d-12	1.1d-14

Interpolation performance for varying band limits, for $\varepsilon = 10^{-7}$, for $e^{i\,c\,a\,x}$

c	n	nodes/λ	Error	N_{Cheb}
5.0	13	8.168	0.12E-06	17
10.0	18	5.655	0.13E-06	24
20.0	26	4.084	0.28E-06	37
30.0	33	3.456	0.73E-06	49
40.0	41	3.220	0.27E-06	61
45.0	44	3.072	0.60E-06	67
50.0	48	3.016	0.33E-06	73
100.0	82	2.576	0.46E-06	128
200.0	147	2.309	0.15E-05	235
300.0	212	2.220	0.17E-05	340
400.0	277	2.176	0.14E-05	443
500.0	341	2.143	0.22E-05	547
1000.0	662	2.080	0.24E-05	1058
1500.0	982	2.057	0.25E-05	1566
2000.0	1301	2.044	0.35E-05	2072

Antenna Design, Prolate Spheroidal Functions

- Far-field radiation pattern of antennas of line sources

$$F(\sin\theta) = \int_{-1}^{1} \sigma(u) \cdot e^{i \cdot k \cdot u \cdot \sin\theta}\, du$$

  where θ is the angle from the normal of the line segment

- Radiation pattern synthesis: given F, find σ!
- Optimal approximation in least square sense is linear combination of first N Prolate functions
- Patterns via discrete arrays of elements

$$\begin{aligned} F(\sin\theta) &= \int_{-1}^{1} \sigma(u) \cdot e^{i \cdot k \cdot u \cdot \sin\theta}\, du \\ &\sim \sum_{j=1}^{n} w_j \cdot e^{i \cdot k \cdot u_j \cdot \sin\theta} \end{aligned}$$

- Looks like a quadrature formula: integrating $e^{i \cdot k \cdot \sin\theta \cdot u}$ with weight σ

$\triangleright$ Use our quadrature machinery!

Example 1: Sector Pattern with 20-wavelength array (k=62.8)

$$\sigma(t) = \frac{sin(k \cdot t)}{t},$$

$$F(\sin\theta) = \int_{-1}^{1} \frac{sin(k \cdot t)}{t} \cdot e^{i \cdot k \cdot t \cdot \sin\theta} \; dt$$

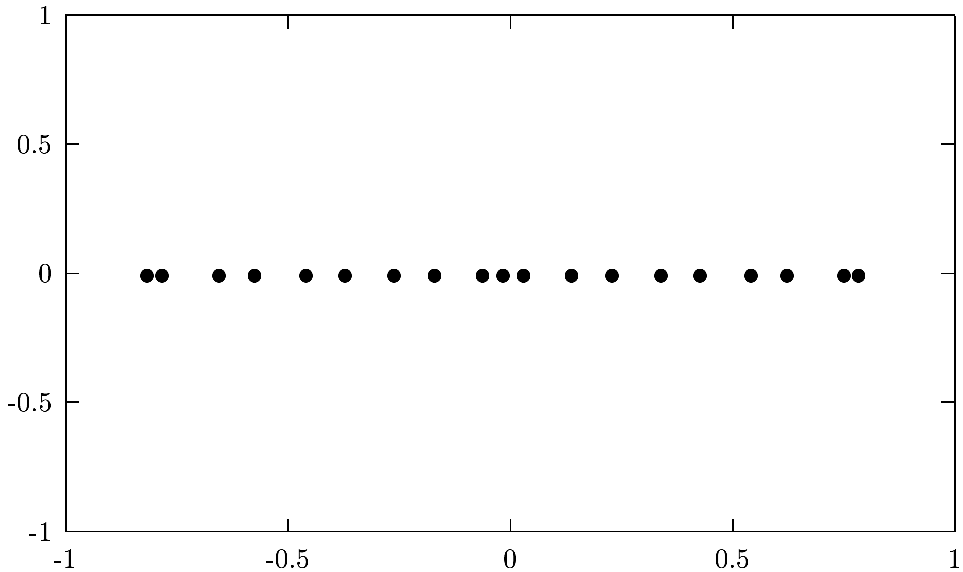


Figure 5a: Configuration generating the pattern in Figure 5

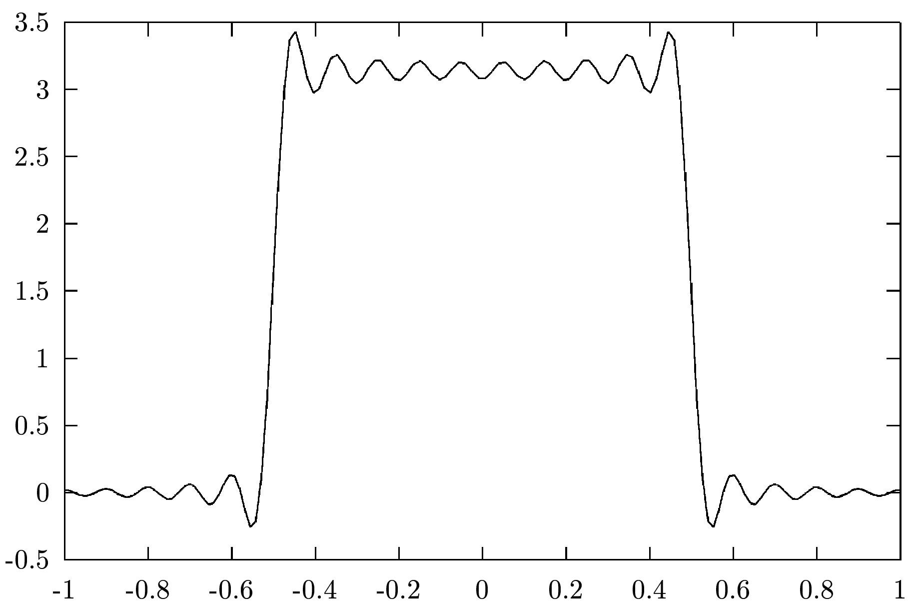


Figure 5: The optimal approximation to the sector pattern with k=62.8

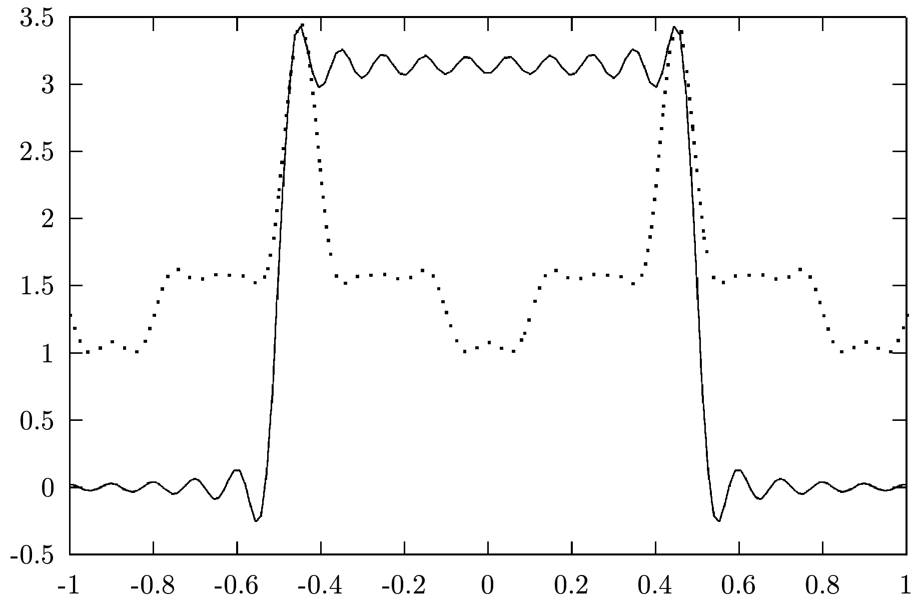


Figure 5b: k=62.8, 19 equispaced nodes

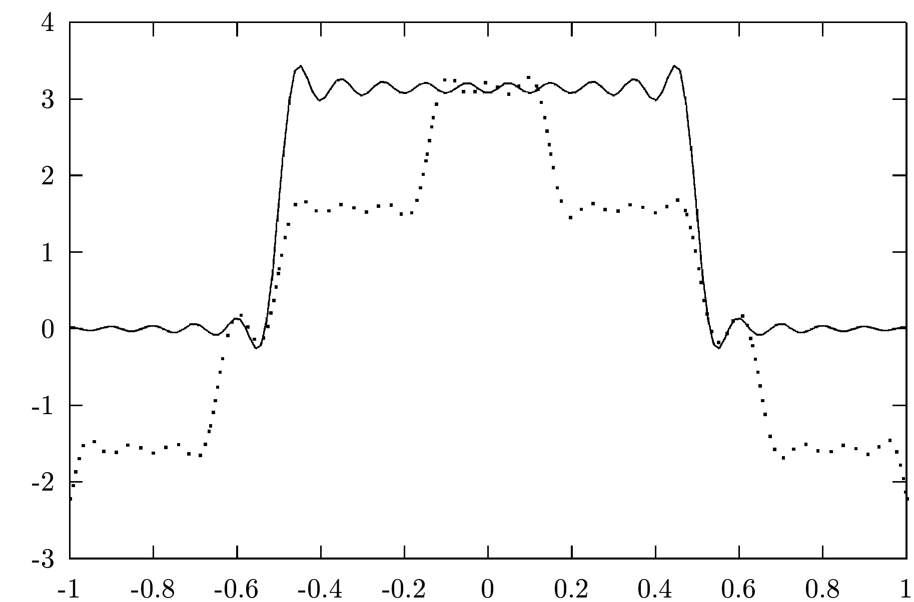


Figure 5c: k=62.8, 24 equispaced nodes

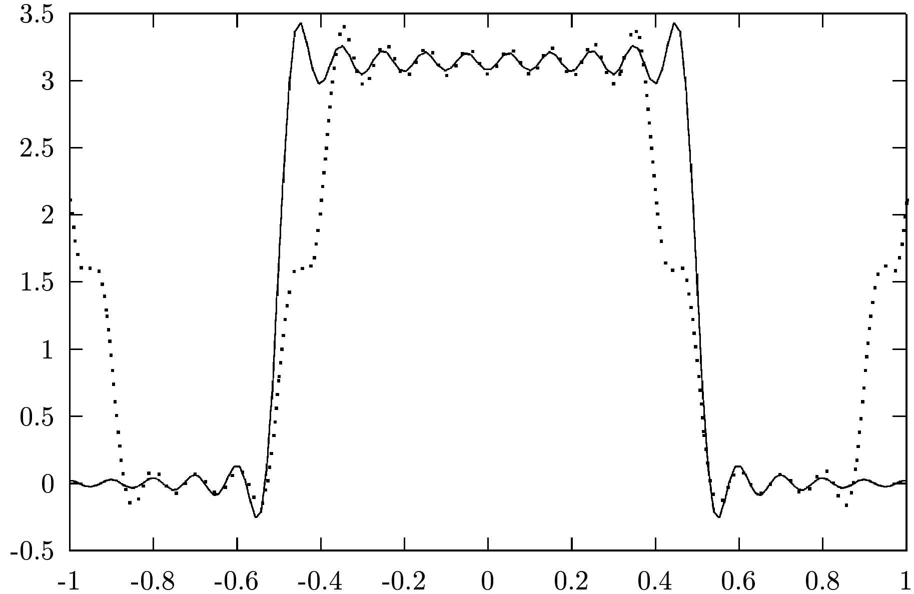


Figure 5d: k=62.8, 29 equispaced nodes

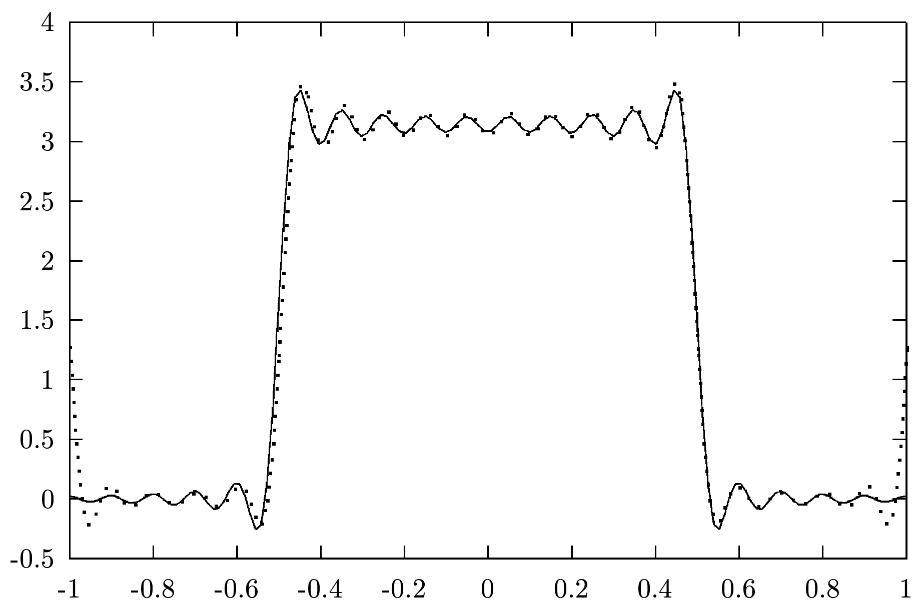


Figure 5e: k=62.8, 31 equispaced nodes

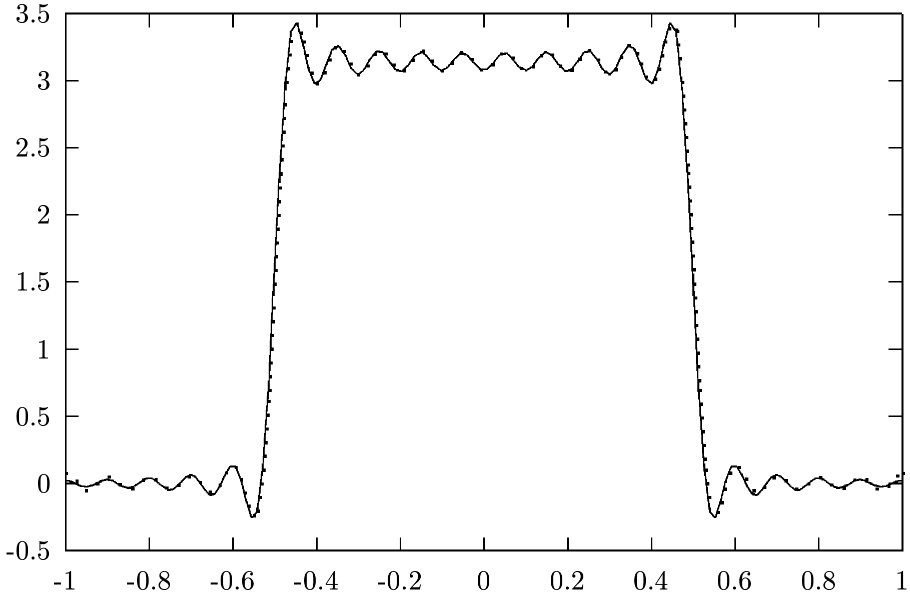


Figure 5f: k=62.8, 34 equispaced nodes

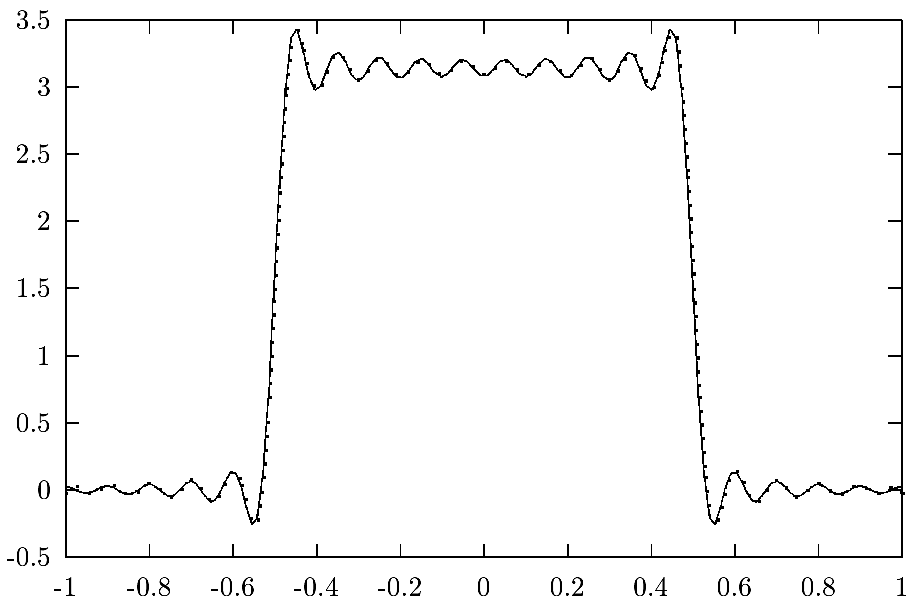


Figure 5g: k=62.8, 21 optimal nodes

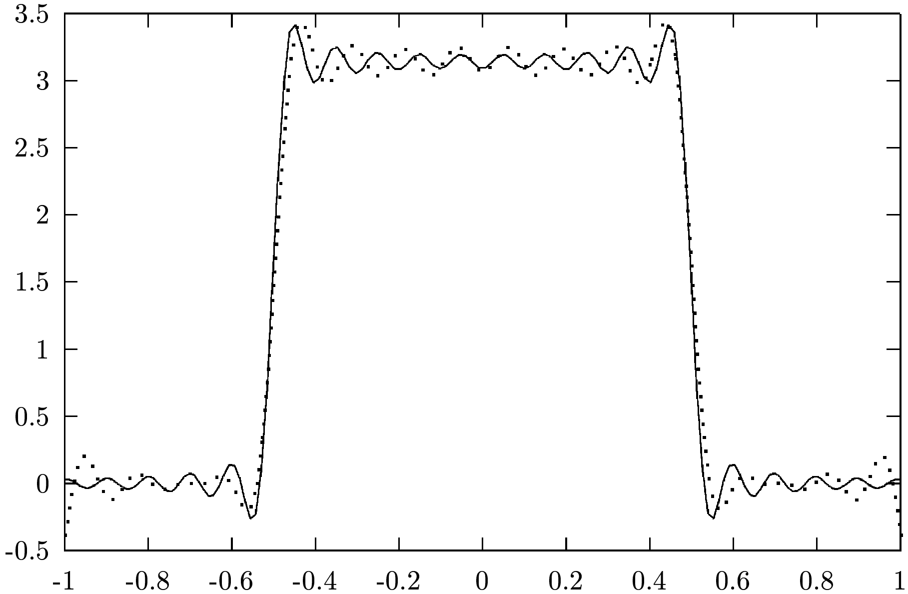


Figure 5h: k=62.8, 17 optimal nodes

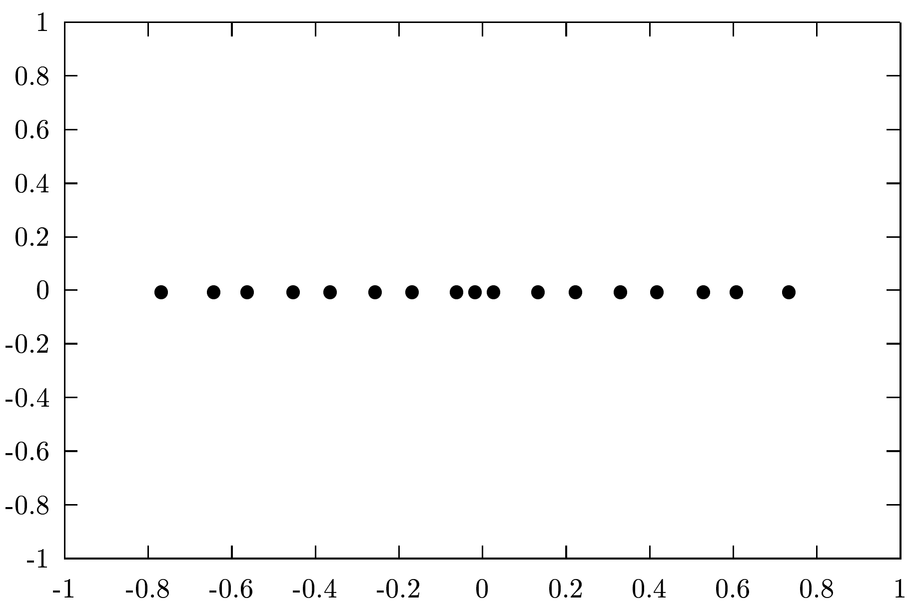


Figure 5i: Configuration generating the pattern in Figure 5h

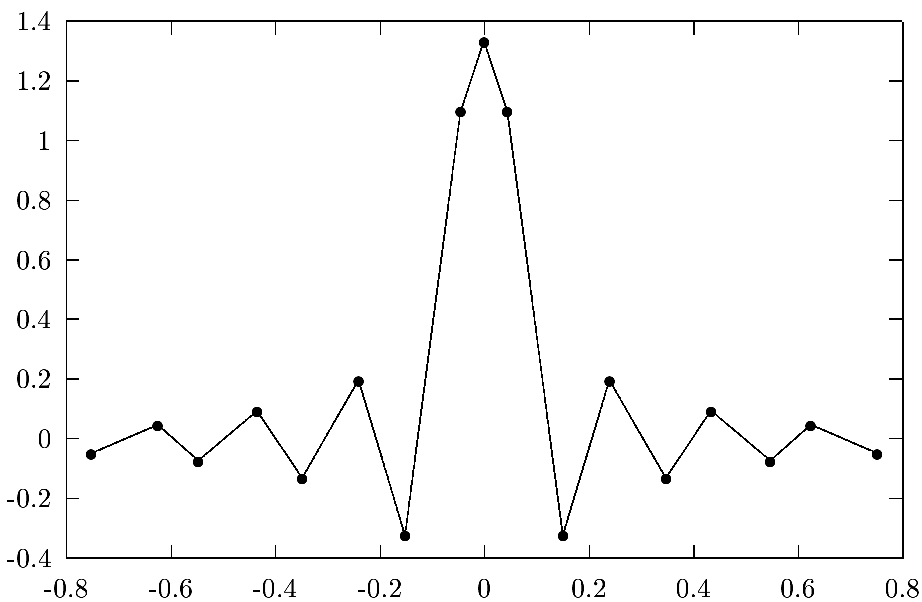


Figure 5j: The values of the 17 sources located at the nodes depicted in Figure 5i and generating the pattern depicted in Figure 5h

Example 2: Cosecant pattern with 35-wavelength antenna array (k=110)

In this example, we set

$$F(x) = \frac{1}{x}$$

for all $x \in [a, b]$, and

$$F(x) = 0$$

for all $x \in \big([-1, 1] \setminus [a, b]\big)$;

$$a = sin(15^\circ),$$

$$b = sin(75^\circ)$$

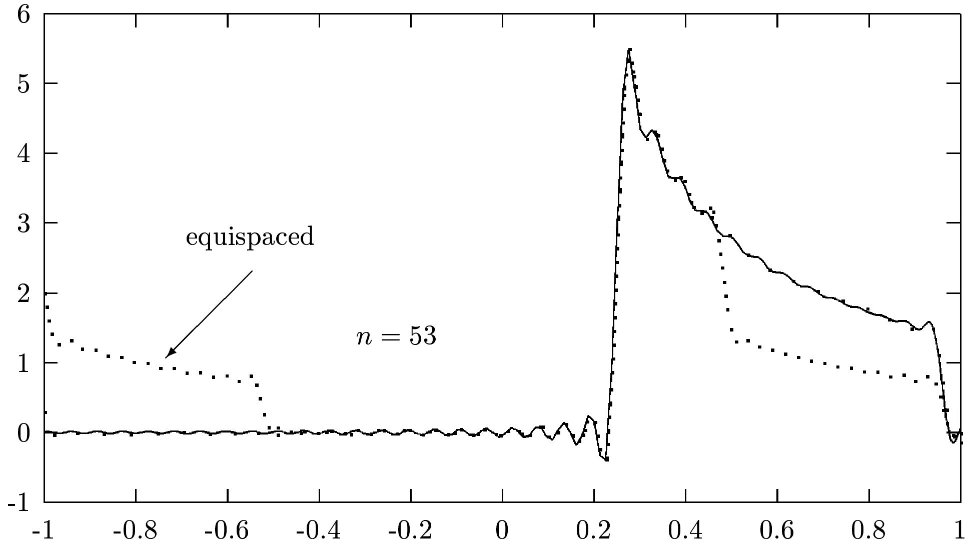


Figure 8a: Cosecant pattern with k=110; n=53

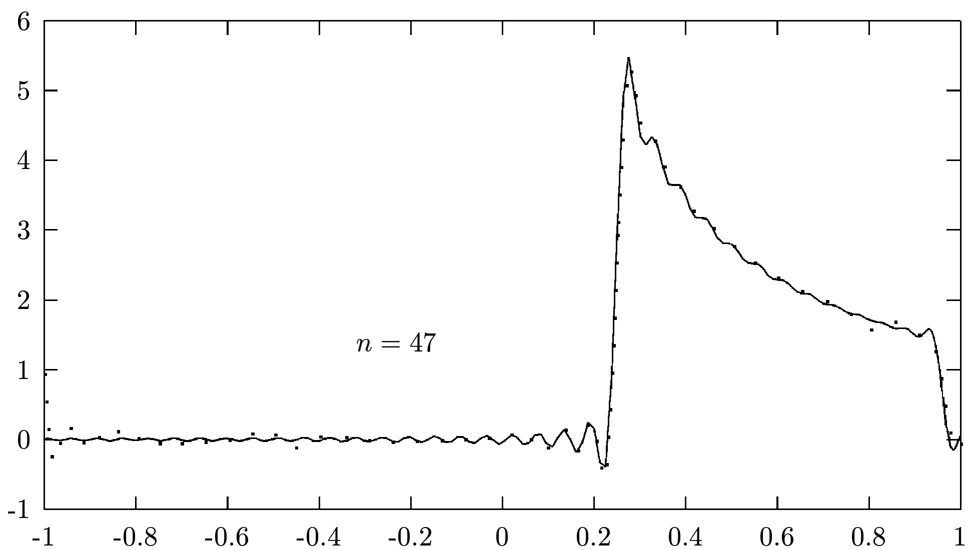


Figure 8b: Cosecant pattern with k=110; n=47. Note: 71 equispaced nodes required to obtain this accuracy

Observations

- $\triangleright$ Comparison with equispaced elements
- $\triangleright$ Improvement of 30% - 50%
- $\triangleright$ Improvement greater when the pattern is symmetric
- $\triangleright$ Gain squared for rectangular arrays
- $\triangleright$ In most cases, σ is not positive

Conclusions

- ▷ Numerical Evaluation of Prolate Spheroidal Wave Functions is straightforward in the current scientific computation environment
- ▷ Quadrature and Interpolation formulae parallel Gaussian quadratures and corresponding interpolation schemes
- ▷ Natural tools for the analysis and numerical computation of band-limited functions
- ▷ Overcome certain limitations of traditional methods in Fourier analysis

Future Work

- ▷ Analysis, Numerical algorithms for approximation, extrapolation with band-limited functions
- ▷ Higher dimensions, disks, rectangles, triangles, spheres
- ▷ Applications of PSWFs in inverse scattering, signal processing, etc.